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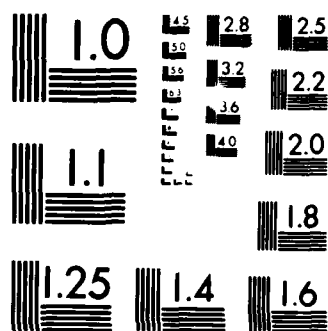
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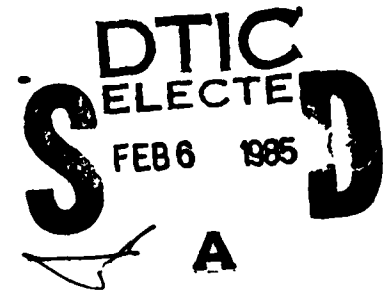
Jack K. Cohen and Frank G. Hagin

Partially supported by the Consortium Project of the
Center for Wave Phenomena and by the Selected Research
Opportunities Program of the Office of Naval Research

Colorado School of Mines

Golden, Colorado 80401

Center for Wave Phenomena
Department of Mathematics
303/273-3557



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**Center for Wave Phenomena
Department of Mathematics
Colorado School of Mines
Golden, Colorado 80401
(303) 273-3557**

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ABSTRACT

The purpose of this work is to present an inversion algorithm for backscattered (^ostacked^o) seismic data which will reconstruct the velocity profile in realistic earth conditions. The basic approach follows that of the original Cohen and Bleistein paper [1979a] in that high frequency asymptotics and perturbation methods are used. However, in the original paper the perturbation was relative to a constant reference speed, whereas the current work uses a reference speed which may vary with depth. This greatly enhances the validity of the perturbation assumption and hence the inversion results. On the other hand, the new algorithm enjoys the same economies and stability properties of the original algorithm, making it very competitive with current migration schemes.

Four major assumptions are made: (i) the acoustic wave equation is an adequate model, (ii) stacked data has amplitude information worth preserving fairly accurately, (iii) the actual reflectivity coefficients can be adequately modeled as perturbations from a continuous reference velocity which depends only on the depth variable, and (iv) the subsurface can be adequately modeled as a series of layers with jump discontinuities in the velocity (or impedance) at these layers.

While the algorithm is particularly suited for data generated by a number of reflecting surfaces, its validity for a single reflector is demonstrated by applying the algorithm to Kirchhoff data for a quite general surface.

A key feature of the approach of this paper is the repeated application of high frequency asymptotic methods; both in obtaining the basic integral equation describing the unknown velocity correction, and in the inversion of this integral equation. Perhaps a noteworthy feature is that the underlying integral equation is in the form of a generalized Fourier integral equation; and the method for its (approximate) inversion may prove to be applicable to a wide range of such problems.

GLOSSARY

A	amplitude of Green's function (B-1)
$b(\underline{r}, \underline{r}')$	a basic amplitude (14)
B	inversion amplitude (10)
c_1	velocity below reflector (30)
$c(z)$	reference velocity (1)
D, E, F, G, H(K, z)	various integrals; see Appendix A
f	frequency (46)
$g(\underline{x}, z; \underline{\xi}, \omega)$	Green's function for $c(z)$ medium (2)
$h(x)$	cylindrical surface (25)
$I(\underline{r}, \underline{r}')$	delta-function-like integral (12)
K	ray parameter (3)
$k_z(K, z)$	$\sqrt{n^2(z) - K^2}$ (6)
$n(z) = c(0)/c(z)$	the index of refraction (7)
$\underline{r} = (x, y, z)$	cartesian (2)
R	reflection coefficient (26)
R_j	reflection coefficient (23)
s, s_j	arclength variables (23)
S	abbreviation; see (29)
$u_S(\underline{\xi}, \omega)$	observed scattered field (2)
$U_S(\underline{\xi}, t)$	observed scattered field (45)
$v(\underline{x}, z)$	velocity (1)
W	the inversion operator (10)
$\underline{x} = (x, y)$	horizontal cartesian (1)
z	vertical cartesian (1)
$\alpha(\underline{x}, z)$	unknown perturbation in velocity (1)

β	reflectivity function (23)
γ	abbreviation; see (26)
γ_1	abbreviation; see (30)
δ	Dirak delta function; below (12)
δ_B, δ_B'	band-limited distributions (18)
Δc	jump in c (31)
$\xi = (\xi, \eta)$	cartesians for observation point at $z = 0$ (2)
ρ	signed distance from \underline{r}' (20)
$\tau(K, z)$	travel time (8)
Φ	phase function (14)
ω	circular frequency (2)

INTRODUCTION

Carter and Frazer [1984], and Bleistein and Gray [1984] (henceforth BG), present inversion algorithms which include the effect of a stratified reference velocity, $c(z)$. Those papers did not address the question of obtaining accurate values of the reflection coefficient; this is the issue treated here. Thus, in the language of Bleistein, Cohen and Hagin [1984], (henceforth BCH), the earlier algorithms provided structural inversions, the location of the sub-surface layers; whereas the present algorithm also provides an accurate estimate of the reflectivity function, which depicts the reflectors and provides an estimate of the reflection strengths across the layers.

Since we employ a perturbation assumption (the "Born Approximation"), the constant reference speed inversion first described in Bleistein and Cohen [1979a] and reviewed in BCH, is often not adequate at depth. Although recursive use of the algorithm is possible and although the results can be significantly enhanced by suitable pre- or post-processing (e.g., see Hagin and Cohen [1984]), extension of the perturbation method to a stratified reference profile is highly significant. It is far more likely that the actual velocity function can be well approximated by a stratified reference velocity than by a constant one, which in turn enhances the validity of the perturbation assumption and the inversion results. See BG for further discussion of this point.

The algorithm presented here has the same structure as the BG algorithm and hence it can be expected to exhibit the same stability and economy. In particular, we note that the processing times for this algorithm with depth-

dependent background velocity will be comparable to those for a constant background k - f algorithm. In addition, we shall show below that the algorithm can be expected to be quite robust even when the "small" perturbation assumption is violated.

A key feature of our approach to this problem is repeated application of high frequency asymptotic methods to obtain an inversion formula valid in the high frequency regime. Discussion of the motivation and justification for such high frequency approximation may be found in BG and BCH. In particular, we shall use a ray theoretic Green's function in formulating our basic integral equation; equation (2) below. A similar approach was presented in Clayton and Stolt [1981]. Moreover, since the resulting integral equation cannot, in general, be inverted exactly, we also use asymptotic methods for this task. This is carried out in the next section. It is perhaps noteworthy that this integral equation can be viewed as a generalized Fourier integral equation; and hence the method of inversion may prove to be of interest outside the present context. In the subsequent section the resulting algorithm is verified (asymptotically) for a quite general class of reflecting surfaces. A short section on computational considerations is included. Finally, many of the detailed calculations are carried out in Appendices A-E.

HIGH FREQUENCY INVERSION

Here we describe the formalism for determination of an asymptotic inversion operator. We employ the same wave equation model as described in detail in BCH. If v is the velocity in the wave equation, we set

$$\frac{1}{v^2(\underline{x}, z)} = \frac{1}{c^2(z)} \left[1 + a(\underline{x}, z) \right], \quad \underline{x} = (x, y). \quad (1)$$

We will also use $\underline{r} = (\underline{x}, z) = (x, y, z)$. Here $c(z)$ is the known stratified reference velocity, while $a(\underline{r})$ is the desired perturbation correction to the actual velocity. Furthermore, we retain the assumption of backscatter ("stacked") data. In this case the basic integral equation for $a(\underline{r})$ is (cf. BCH, equation (8)):

$$u_S(\underline{\xi}; \omega) = \omega^2 \int \int \int d^3 r \frac{a(\underline{r})}{c^2(z)} g^2(\underline{r}; \underline{\xi}; \omega), \quad \underline{\xi} = (\xi, \eta) \quad (2)$$

where all unmarked integral signs are over $(-\infty, \infty)$. Here u_S denotes the backscattered field at the location $\underline{\xi} = (\xi, \eta)$ on the observation plane, $z = 0$, and g (the "incident field") denotes the Green's function corresponding to the stratification, $c(z)$. In contrast to the constant background case, g cannot be determined exactly; we must use the high frequency assumption. Fortunately, this assumption is completely justified on the geophysical exploration scale and has long been used to simplify processing formulas even when it was possible to derive wide band analytic results (see BCH). We use J. B. Keller's [1978] ray method formalism (see also Bleistein [1984]), which is the multi-dimensional analogue of the WKB method to obtain a parametric representation of g (see Appendix B):

$$g \sim \frac{e^{i\omega\tau(K, z)}}{4\pi \sqrt{k_3(K, 0) k_3(K, z) E(K, z) H(K, z)}} . \quad (3)$$

Here, if we introduce the transverse distance,

$$|\underline{x} - \underline{\xi}| , \quad (4)$$

then the parameter, K , in (3) is defined as a function of $|\underline{x} - \underline{\xi}|$ and z by the ray equation:

$$|\underline{x} - \underline{\xi}| = K E(K, z) . \quad (5)$$

Further, the quantity $k_3(K, z)$ is given by

$$k_3(K, z) = \sqrt{n^2(z) - K^2} , \quad (K^2 < n^2(z)) \quad (6)$$

where in turn, n , the index of refraction, is

$$n(z) = \frac{c(0)}{c(z)} . \quad (7)$$

The travel-time, τ , is

$$\tau = \frac{1}{c(0)} G(K, z) , \quad G = \int_0^z \frac{n^2(\xi) d\xi}{k_3(K, \xi)} , \quad (8)$$

and finally, the quantities E and H are likewise integrals involving n and k_3 . These integrals, as well as others that occur subsequently, are defined in Appendix A. There, we also derive some needed relations involving these quantities.

We shall not need the extension of k_z in (6) to the range $n^2(z) < K^2$ because our Fourier transforms are integrals over real wave numbers only.

Thus our task is to invert the integral equation,

$$u_S(\underline{\xi}, \omega) = \frac{\omega^2}{16\pi^2} \int \int \int d^3r \frac{a(\underline{r})}{c^2(z)} \frac{e^{2i\omega\tau(K, z)}}{k_z(K, 0) k_z(K, z) E(K, z) H(K, z)} \quad (9)$$

for $a(\underline{r})$ in terms of the data, $u_S(\underline{\xi}, \omega)$. Again, the ray parameter K is defined by (5).

Since the phase in (9) resembles that of a Fourier transform, we are motivated to seek an asymptotic inversion operator W of the form:

$$W[u_S(\underline{\xi}, \omega)](\underline{r}') \sim \int \int d^2\xi \int d\omega B(\underline{r}', \underline{\xi}) e^{-2i\omega\tau(K', z')} u_S(\underline{\xi}, \omega) \quad (10)$$

where the amplitude $B(\underline{r}', \underline{\xi})$ and value K' must be determined. Condition (5) above suggests that K' satisfies $|\underline{r}' - \underline{\xi}| = K'E(K', z)$. In (10) we have introduced primes to avoid confusion with the integration variables in (9). Applying W to both sides of (9) and writing out the right side explicitly we have:

$$a(\underline{r}') = W[u_S](\underline{r}') \sim \frac{1}{16\pi^2} \int d\omega \int \int d^2\xi \int \int \int d^3r$$

$$\frac{a(\underline{r})}{c^2(z)} \omega^2 B(\underline{r}, \underline{\xi}') \frac{\exp\{2i\omega[\tau(K, z) - \tau(K', z')]\}}{k_z(K, 0) k_z(K, z) E(K, z) H(K, z)} \quad (11)$$

$$= \int \int \int d^3r a(\underline{r}) I(\underline{r}, \underline{r}'), \quad \text{where}$$

$$I(\underline{r}, \underline{r}') = \frac{1}{16\pi^2} \int d\omega \omega^2 \int \int d^2\xi \frac{B(\underline{r}', \underline{\xi}) e^{2i\omega[\tau(K, z) - \tau(K', z')]} }{c^2(z) k_z(K, 0) k_z(K, z) E(K, z) H(K, z)} . \quad (12)$$

Clearly, if (11) is to hold, then $I(\underline{r}, \underline{r}')$ must represent the 3-dimensional delta function, $\delta(\underline{r} - \underline{r}')$. Hence our task is to find $B = B(\underline{r}', \underline{\xi})$ and K' , so that this is the case.

First a comment about the assumed form of B , i.e., $B(\underline{r}', \underline{\xi})$ where $\underline{r} = (x, y, z)$ and $\underline{\xi} = (\xi, \eta)$. In attempting to invert (9) for $a(\underline{r}')$ one would generally expect B to also involve ω . However, our experience with canonical problems (e.g. $c(z) = \text{constant}$) suggests that B is independent of ω , and the work to follow confirms this.

In (11) and (12) we must acknowledge that in Geophysical applications ω is confined to the high frequency regime (e.g. see BCH for details). Hence we may evaluate the asymptotic (large ω) approximation to the $\underline{\xi}$ integral in (12) by stationary phase. For convenience we will think of ω as the "large parameter" in our asymptotics to follow. More precisely, it is important

that the dimensionless parameter $2\omega L/c(0)$ be "large", where L is a typical length scale of the problem. This is discussed more fully in BG. The details of the stationary phase calculation are carried out in Appendix D. The conclusions are:

- 1) $K' = K$. Geometrically, this stationarity condition says that for given \underline{r} and \underline{r}' , $\xi_s = \xi_s$ is chosen so that \underline{r} , \underline{r}' and $(\xi_s, 0)$ lie on the same ray. See Fig. 1. Since $K' = K$ we will use only K hereafter. Hence, along with (5), we now have

$$|\underline{r}' - \xi| = KE(K, z') \quad . \quad (13)$$

- 2) The asymptotic approximation to (12) is as follows where $E' = E(K, z')$, $H' = H(K, z')$ and ξ_s symbolizes that the stationarity condition, $K' = K$, is to be applied:

$$I(\underline{r}, \underline{r}') \sim \frac{1}{16\pi^2} \int d\omega \omega^2 \left[\frac{i}{\omega} b(\underline{r}, \underline{r}') e^{2i\omega[\tau(K, z) - \tau(K, z')]} \right] ,$$

where

$$b(\underline{r}, \underline{r}') = \frac{B \pi c(0) \operatorname{sgn}(z' - z) (E'H'/EH)^{1/2}}{c^2(z) k_z(K, 0) k_z(K, z) [(E - E')(H - H')]^{1/2}} \quad . \quad (14)$$

This expression for $I(\underline{r}, \underline{r}')$ will simplify dramatically after we perform the ω integration. At this point we have

$$I(\underline{x}, \underline{x}') \sim \frac{b(\underline{x}, \underline{x}')}{16\pi^2} \int d\omega \, i\omega \, e^{2i\omega[\tau(K, z) - \tau(K, z')]} \quad (15)$$

If we denote $\tau' = \tau(K, z')$, those familiar with distributions will recognize the above ω integral as the distribution

$$\frac{\pi}{2} \delta'(\tau - \tau') = \frac{\pi}{2} \frac{d}{d\tau} \delta(\tau - \tau') \quad (16)$$

where δ is the standard delta function. The distribution in (16) has the sifting property that, for any differentiable $f(\tau)$,

$$\int_{-\infty}^{\infty} f(\tau) \delta'(\tau - \tau_0) d\tau = -f'(\tau_0) \quad (17)$$

We will use this property shortly. We now have

$$I(\underline{x}, \underline{x}') \sim \frac{b(\underline{x}, \underline{x}')}{32\pi} \delta_B'(\tau - \tau') \quad (18)$$

where the subscript B serves as a reminder that, in application of these results the frequency ω will actually be band-limited. Hence the ω integral in (15) is proper and, consequently, the resulting distribution δ' is in fact a band-limited version (i.e., a regular function). These matters have been discussed in Cohen and Bleistein [1979b], Bleistein [1984] and BCH [1984]. We will use the notation δ_B and δ_B' for this reason, but proceed as

if all the action in $I(\underline{r}, \underline{r}')$ takes place as $\underline{r} \rightarrow \underline{r}'$.

If one turns to $b(\underline{r}, \underline{r}')$ defined by (14) and refers to the definitions in Appendix A, it is easy to show that as \underline{r} approaches \underline{r}' ,

$$b(\underline{r}, \underline{r}') \sim \frac{B k_3(K, z')}{32 c(z') k_3(K, 0) (z' - z)} .$$

Using this in (18) gives

$$I(\underline{r}, \underline{r}') \sim \frac{B k_3(K, z')}{32 c(z') k_3(K, 0) (z' - z)} \delta_B'(\tau - \tau') . \quad (19)$$

Notice the singularity $(z' - z)^{-1}$, as $z \rightarrow z'$, in this form of I . This is precisely the amount of singularity needed, when combined with $\delta'(\tau - \tau')$, to produce the required 3-dimensional distribution $\delta(\underline{r} - \underline{r}')$. Recall that as $z \rightarrow z'$, $\underline{r} \rightarrow \underline{r}'$ along the common ray and $\tau = \tau(K, z) \rightarrow \tau(K, z') = \tau'$. In Appendix E the following is established, if

$\rho \equiv |\underline{r} - \underline{r}'| \operatorname{sgn}(z - z')$, then

$$\frac{d}{d\tau} \delta(\tau - \tau') = c^2(z') \frac{d}{d\rho} \delta(\rho) = c^2(z') \delta'(\rho),$$

$$\lim_{z \rightarrow z'} \frac{\rho}{z - z'} = \frac{n(z')}{k_z(K, z')}.$$

We use these two results to rewrite the singular portion of (19), dropping the subscript B for now,

$$\frac{\delta'(\tau - \tau')}{z' - z} = \frac{n(z')}{k_z(K, z')} \frac{\delta'(\rho)}{-\rho}. \quad (20)$$

We now verify that $-\delta'(\rho)/\rho$ behaves like $2\pi \delta(\underline{r} - \underline{r}')$ by integrating it against an arbitrary differentiable function $f(\underline{r})$ over an ε -sphere centered at \underline{r}' .

$$\begin{aligned} \int \int \int_{|\underline{r} - \underline{r}'| \leq \varepsilon} dV f(\underline{r}) \frac{\delta'(\rho)}{-\rho} &= \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi \int_0^\varepsilon d\rho \rho^2 f(\underline{r}) \frac{\delta'(\rho)}{-\rho} \\ &= \int_0^{\pi/2} d\theta \sin \theta \int_0^{2\pi} d\phi \int_{-\varepsilon}^\varepsilon d\rho [\rho f(\underline{r})] \frac{\delta'(\rho)}{-1} \\ &= 2\pi [\rho f(\underline{r})]' \Big|_{\rho=0} = 2\pi f(\underline{r}'). \end{aligned}$$

Note, in the second step, the compensating change of limits in the θ and ρ integrals. In the last step we used the sifting property (17) of $\delta'(\rho)$. In the sense of 3-dimensional distributions, we have established

$$\frac{\delta'(\rho)}{-\rho} = 2\pi\delta(\underline{x}-\underline{x}') .$$

Using this and (20) in (19), and returning to our δ_B notation, we have at last

$$I(\underline{x}, \underline{x}') \sim \frac{B \pi c(0)}{16 k_s(K, 0)} \delta_B(\underline{x}-\underline{x}') .$$

For this to represent $\delta_B(\underline{x}-\underline{x}')$ we are led to define amplitude B as

$$B = \frac{16 k_s(K, 0)}{\pi c(0)} = \frac{16 \sqrt{1-K^2}}{\pi c(0)} . \quad (21)$$

This completes the definition of our inversion operator W in (10). Applying W to (9) gives our inversion for $a(\underline{x}')$,

$$a(\underline{x}') \sim \frac{16}{\pi c(0)} \int \int d^2 \xi \int d\omega \sqrt{1-K^2} e^{-2i\omega\tau(K, z')} u_S(\underline{\xi}, \omega) \quad (22)$$

where K is defined implicitly by

$$|\underline{x}' - \underline{\xi}| = K E(K, z') = K \int_0^{z'} \frac{d\tau}{\sqrt{n^2(\tau) - K^2}} .$$

As discussed in BCH, once we surrender knowledge of the low frequency input information, we cannot obtain output trend information. It is to be hoped that (by iteration if necessary) our $c(z)$ reference velocity is an adequate approximation of the trend to the depths of interest. What we can obtain from band-limited information is a perturbation correction consistent with the model of jumps across a series of interfaces. We determine the approximate location of these interfaces as well as the approximate value of the reflection coefficient at the interfaces. This information is summed up in the reflectivity function,

$$\beta = \sum R_j \delta_B(s_j) \quad (23)$$

where s_j is a (local) arclength variable measured normally from the j^{th} interface and R_j is the normal reflection coefficient of that interface. Clearly, knowledge of β is equivalent to knowledge of reflector location and the normal reflection coefficient (see equation (44) below). In turn the latter allows direct computation of the jump in c across the reflector.

According to the theory developed in Cohen and Bleistein (1979b) and reviewed in BCH, we can obtain β from α by inserting a factor of $i\omega/2c(z)$ in (22) to obtain (after dropping the primes):

$$\beta(\underline{r}) \sim \frac{8i}{\pi c(0)c(z)} \int \int d^2\xi \sqrt{1-K^2} \int d\omega \omega e^{-2i\omega\tau(K,z)} u_S(\underline{\xi}, \omega) \quad . \quad (24)$$

Verification of Algorithm

In order to verify the feasibility of the algorithm in (24), we first obtain an expression for the Kirchhoff representation of data u_S . Such data employs the high frequency assumption (by using the multi-dimensional WKB representation of the incident, reflected and transmitted fields), but does not make the Born approximation of small reflection coefficient. Thus, if we can show asymptotic equality in (24) for such data, our algorithm is likely to be quite robust for large contrast interfaces.

It remains to decide on the surfaces to use in computing the Kirchhoff approximation to u_S . For ease of presentation, we carry out our calculations for the general cylindrical (i.e. y independent) surface:

$$z = h(x) \quad . \quad (25)$$

With a little more effort the verification can be done for a quite general reflecting surface.

In Appendix C, we show that the back scatter at ξ from (25) has the Kirchhoff (high frequency) representation:

$$u_S(\xi, \omega) \sim 2i\omega \int \int d^2 \bar{x} \sqrt{1+h'^2(\bar{x})} \gamma RS e^{2i\omega\tau(\bar{K}, \bar{z})} \quad (26)$$

subject to

$$|\bar{x} - \xi| = \bar{K} E(\bar{K}, \bar{z}) \quad , \quad \bar{z} = h(\bar{x}) \quad . \quad (27)$$

Here we have used bars to distinguish the spatial variable from the output

variables in the inversion formula (24) and have also introduced the quantities:

$$\gamma = \frac{\left[\frac{(\bar{x}-\xi)h'}{E(\bar{K}, \bar{z})} - k_3(\bar{K}, \bar{z}) \right]}{c(0) \sqrt{1+h'^2}}, \quad (28)$$

$$S = \frac{1}{16\pi^2 k_3(\bar{K}, 0) k_3(\bar{K}, z) E(\bar{K}, z) H(\bar{K}, z)}, \quad (29)$$

and the (non-normal) reflection coefficient,

$$R = \frac{\gamma - \gamma_1}{\gamma + \gamma_1}, \quad \gamma_1 = \operatorname{sgn}(\gamma) \sqrt{\gamma^2 + \frac{1}{c_1^2} - \frac{1}{c^2}}. \quad (30)$$

In turn, $c_1(z)$ denotes the actual velocity below the reflector, $z = h(\bar{x})$, that is,

$$c_1(z) = c(z) + \Delta c \quad (31)$$

where Δc is not accounted for by the stratified reference profile $c(z)$. Obviously determining R is tantamount to determining Δc or $c_1(z)$. Putting data, u_g given by (26), into (24) leads to

$$\beta(\underline{x}) \sim \frac{-16}{\pi c(0)c(z)} \int d\omega \omega^2 \int \int d^2 \xi \int \int d^2 \bar{x} \cdot \sqrt{1-K^2} \sqrt{1+h'^2} \gamma R S e^{2i\omega[\tau(\bar{K}, \bar{z}) - \tau(K, z)]}. \quad (32)$$

Our goal is to show that the output of this expression does indeed represent the reflecting surface.

We now carry out stationary phase in \bar{x} , \bar{y} , ξ , η (see Appendix D for details) and obtain the stationarity conditions:

$$\bar{y} = \eta = y \quad , \quad (33)$$

$$\bar{x} - \xi = -\text{sgn}(h') \bar{K} E(\bar{K}, \bar{z}) \quad , \quad (34)$$

and

$$x - \xi = -\text{sgn}(h') K E(K, z) \quad . \quad (35)$$

These imply:

$$K = \bar{K} = \frac{n(\bar{z}) |h'|}{\sqrt{1+h'^2}} \quad , \quad (36)$$

$$k_z(\bar{K}, \bar{z}) = \frac{n(\bar{z})}{\sqrt{1+h'^2}} \quad , \quad (37)$$

where again $\bar{z} = h(\bar{x})$.

Geometrically, these conditions confirm the cylindrical nature of our reflector and show that the output point (x, z) lies on a specular ray. Furthermore (33-35) yield

$$\gamma = -\frac{1}{c} \quad , \quad \gamma_1 = -\frac{1}{c_1} \quad (38)$$

which, in turn, imply that R reduces to the normal reflection coefficient,

$$R = R_n = \frac{c_1 - c}{c_1 + c} \quad (39)$$

Completing the stationary phase analysis (again see Appendix D) we find that

$$\beta(\underline{x}) \sim \frac{16\pi n^2(z) \sqrt{1+h'^2} R_n S \sqrt{1-K^2}}{c(0) \sqrt{|\det(\Phi_{ij})|}} \cdot \int d\omega e^{2i\omega[\tau(\bar{K}, \bar{z}) - \tau(\bar{K}, z)]} \quad (40)$$

Here,

$$\det(\Phi_{ij}) = \frac{1}{E(\bar{K}, z)E(\bar{K}, \bar{z})} \left[\frac{(1+h'^2)^2}{H(\bar{K}, z)H(\bar{K}, \bar{z})} + \frac{n(z)h''}{\sqrt{1+h'^2}} \left[\frac{1}{H(\bar{K}, z)} - \frac{1}{H(\bar{K}, \bar{z})} \right] \right] \quad (41)$$

However, the final integration in ω , yields a delta function whose argument can be transformed (using the relationships in Appendix A) to arclength along rays:

$$\begin{aligned} \int d\omega e^{2i\omega[\tau(\bar{K}, \bar{z}) - \tau(\bar{K}, z)]} &= \pi \delta_B[\tau(\bar{K}, \bar{z}) - \tau(\bar{K}, z)] \\ &= \pi c(0) \delta_B[G(\bar{K}, \bar{z}) - G(\bar{K}, z)] \\ &= \pi c(0) \frac{k_s(\bar{K}, z)}{n^2(z)} \delta_B[z - \bar{z}] \\ &= \pi c(0) \cdot \frac{1}{n(z)} \delta_B[s(z) - s(\bar{z})] \end{aligned} \quad (42)$$

Here the last equality involving the ray arclength variable follows from (A-8) and Appendix B. We may note that since $z = \bar{z}$ when the delta function "acts," the stationarity conditions (33-35) imply that the output point (\underline{x}, z) coincides with the specular point $(\underline{x}, z) = (\underline{x}, h(\underline{x}))$. Furthermore,

when $z = \bar{z}$ the second term in the large square brackets in (41) drops out and hence using the definition (29), we see that

$$\frac{S}{\sqrt{|\det(\Phi_{ij})|}} = \frac{1}{16\pi^2(1+h'^2)k_z(\bar{K}, 0)k_z(\bar{K}, \bar{z})} \quad (43)$$

Hence (40) reduces to

$$\begin{aligned} \beta(\underline{r}) &\sim \frac{R_n(z)}{\sqrt{1+h'^2} k_z(\bar{K}, \bar{z})} \delta[s(z)-s(\bar{z})] \\ &= R_n \delta[s(z)-s(\bar{z})] \end{aligned} \quad (44)$$

Here, the last step follows from (37).

In summary, when the Kirchhoff data u_S for a single reflecting surface $\bar{z} = h(\bar{x})$ is put into (24) and the computations are carried out asymptotically, the inversion algorithm faithfully reproduces the surface. Moreover, if a linear combination of such data, representing several such surfaces, were inserted into (24) then, in principle, the algorithm could reproduce an appropriate sum of responses as in (23). However, since the background $c(z)$ would presumably not be exact beneath the first reflector, there would be some distortion introduced into the second and subsequent reflectors. This issue was discussed in Hagin and Cohen, 1984.

REMARKS ON DATA PROCESSING

The algorithm for the reflectivity function derived in the previous sections is

$$\beta(\underline{x}, z) \sim \frac{8i}{\pi c(0) c(z)} \int \int d^2 \xi \sqrt{1-K^2} \int d\omega \omega e^{-2i\omega\tau(K, z)} \cdot \int_0^\infty dt U_S(t, \xi) e^{i\omega t} , \quad (45)$$

$$|\underline{x} - \underline{\xi}| = KE(K, z) .$$

Here U_S is the backscatter time data observed on an areal array. For actual data processing, it is convenient to "fold" the unphysical negative frequencies onto the positive ones by replacing ω by $-\omega$ on the interval $(-\infty, 0)$. At the same time, we introduce the physical frequency variable (measured in Hz):

$$f = \frac{\omega}{2\pi} \quad (46)$$

and explicitly acknowledge the bandlimiting by introducing $F(f)$, a tapered high pass filter. After these changes, we have:

$$\begin{aligned}
\beta(\underline{x}, z) &\sim \frac{-64\pi}{c(0)c(z)} \iint d^2\xi \sqrt{1 - K^2} \\
&\cdot \operatorname{Im} \int_0^\infty df f F(f) e^{-4\pi i f \tau(K, z)} \\
&\cdot \int_0^\infty dt U_S(t, \underline{\xi}) e^{2\pi i f t} , \\
|\underline{x} - \underline{\xi}| &= KE(k, z) .
\end{aligned}
\tag{47}$$

In practice, areal observations are often not available and instead only a linear set of data is used. In this case we cannot hope to reconstruct a three dimensional image of the subsurface and instead seek a two dimensional slice, $\beta(x, 0, z) = \beta(x, z)$, consistent with the data available. Since the data is now independent of η ,

$$U_S(t, \underline{\xi}) = U_S(t, \xi, 0) = U_S(t, \xi) , \tag{48}$$

and we may carry out an additional stationary phase calculation in η . The stationarity condition is

$$\eta = y \tag{49}$$

and the analogue of (45) is found to be (see Appendix D for details):

$$\begin{aligned}
\beta(x, z) \sim & \frac{8}{\sqrt{\pi c(0)} c(z)} \int d\xi \sqrt{(1-K^2) E(K, z)} \cdot \\
& \int d\omega \sqrt{i\omega} e^{-2i\omega\tau(K, z)} \\
& \int_0^\infty dt U_S(t, \xi) e^{i\omega t} , \\
|x-\xi| = & K E(K, z) ,
\end{aligned} \tag{50}$$

while (47) becomes:

$$\begin{aligned}
\beta(x, z) \sim & \frac{32\pi}{\sqrt{c(0)} c(z)} \int d\xi \sqrt{(1-K^2) E(K, z)} \cdot \\
& (\text{Re} - \text{Im}) \int_0^\infty df \sqrt{f} F(f) e^{-4\pi if\tau(K, z)} \\
& \cdot \int_0^\infty dt U_S(t, \xi) e^{2\pi ift} , \\
|x-\xi| = & K E(K, z) .
\end{aligned} \tag{51}$$

The basic concepts of reducing (51) to a computer code are the same as those discussed in BG for the algorithm presented there. Briefly, the t and f integrals are performed routinely using an efficient FFT algorithm. The main complication in (51) lies in the expressions $E(K, z)$ and $\tau(K, z)$, both being integrals defined by (A-4) and (8) respectively. This is a bit subtle in that the parameter K (see Appendix B) can be viewed as determining the starting angle for a ray connecting the surface point $(\xi, 0)$ to data point (x, z) in (51). Therefore, for a given offset $|x-\xi|$, K is defined by the implicit relation $|x-\xi| = K E(K, z)$. In computation this issue is handled

quite efficiently by two tables for evaluating $\tau(K, z)$ and the amplitude (involving E in (51)) as functions of $|x - \xi|$ and z .

The computation time of the resulting algorithm, as pointed out in BG, is comparable to a standard k-f migration algorithm with constant reference speed.

CONCLUSIONS

We have presented the derivation of an inversion algorithm for backscattered ("stacked") seismic data. We made four major assumptions: (i) the acoustic wave equation is an adequate model, (ii) stacked data has amplitude information worth preserving fairly accurately, (iii) the actual reflectivity coefficients can be adequately modeled as perturbations from a continuous reference velocity which depends only on the depth variable, (iv) the subsurface can be adequately modeled as a series of layers with jump discontinuities in the velocity (or impedance) at these layers.

The last assumption is unavoidable given the nature of the high pass (on the exploration scale) data collected in the field. The third assumption is inherent in our approach although, as pointed out above, the algorithm can be expected to be robust even when this assumption is violated. Also the algorithm presented here represents a considerable improvement over earlier algorithms, such as Cohen and Bleistein [1979a], which perturbed from a constant reference velocity.

On the other hand, weakening of the first two assumptions seems eminently feasible and we hope to apply the techniques expanded in this article to both inversion of offset data ("inversion before stack") and to equations which more accurately describe the wave propagation in the earth.

It is already known (see BG) that algorithms with the structure of the one presented here are numerically stable and are computationally efficient relative to other seismic data processing algorithms.

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APPENDIX A
NOTATIONS AND IDENTITIES

We define

$$n(z) = \frac{c(0)}{c(z)} \quad , \quad (A-1)$$

$$k_z(K, z) = \sqrt{n^2(z) - K^2} \quad , \quad (A-2)$$

and the integrals,

$$D(K, z) = \int_0^z k_z(K, \xi) d\xi \quad , \quad (A-3)$$

$$E(K, z) = \int_0^z \frac{d\xi}{k_z(K, \xi)} \quad , \quad (A-4)$$

$$F(K, z) = \int_0^z \frac{d\xi}{k_z^3(K, \xi)} \quad , \quad (A-5)$$

$$G(K, z) = \int_0^z \frac{n^2(\xi) d\xi}{k_z(K, \xi)} = c(0)\tau(K, z) \quad (A-6)$$

$$H(K, z) = \int_0^z \frac{n^2(\xi) d\xi}{k_3^3(K, \xi)} , \quad (A-7)$$

$$s(K, z) = \int_0^z \frac{n(\xi) d\xi}{k_3(K, \xi)} . \quad (A-8)$$

Similar quantities occur in the τ -p theory, see Diebold and Stoffa [1981]. Among the many relations which link these quantities, we cite below those that are useful in carrying out the calculations presented in this paper and its appendices.

First of all, from (A-2) it follows that

$$D + K^2 E = G , \quad (A-9)$$

$$E + K^2 F = H . \quad (A-10)$$

Next we cite the k and z partial derivatives of D , E and G which follow respectively from use of

$$\frac{\partial k_3}{\partial K} = - \frac{K}{k_3} \quad (A-11)$$

and the Fundamental Theorem of calculus:

$$D_K = - K E , \quad D_z = k_3(K, z) , \quad (A-12)$$

$$E_K = KF \quad , \quad E_z = \frac{1}{k_j} \quad , \quad (A-13)$$

$$G_K = KH \quad , \quad G_z = \frac{n^2}{k_j} \quad . \quad (A-14)$$

Finally, from (A-13) and (A-9) it follows that

$$(KE)_K = H \quad . \quad (A-15)$$

APPENDIX B

THE STRATIFIED MEDIA GREEN'S FUNCTION

Using Keller's [1978] "ray method", developed in the 1950's, we seek a high frequency approximation,

$$g(\underline{x}, z) \sim A(\underline{x}, z) e^{i\omega\tau(\underline{x}, z)} \quad , \quad \underline{x} = (x, y) \quad (B-1)$$

which asymptotically satisfies the Helmholtz equation,

$$\nabla^2 g + \frac{\omega^2}{c^2(z)} g = -\delta(\underline{x}-\underline{\xi}) \delta(z) \quad , \quad \underline{\xi} = (\xi, \eta) \quad (B-2)$$

To complete the specification of g , we insist that it behave like the free space (i.e. constant c) Green's function as the field point, (\underline{x}, z) , approaches the source point, $(\underline{\xi}, 0)$. This entails the conditions,

$$\tau \rightarrow R/c(0) \quad , \quad A \rightarrow \frac{1}{4\pi R} \quad (B-3)$$

as $R \rightarrow 0$, where

$$R^2 = |\underline{x}-\underline{\xi}|^2 + z^2 \quad (B-4)$$

We substitute (B-1) into (B-2) and separately equate the coefficients of ω^2 and ω to zero (this is the high frequency approximation) giving rise to the eikonal equation,

$$\underline{k} \cdot \underline{k} = n^2(z) \quad , \quad \underline{k} = c(0) \nabla\tau \quad , \quad n = \frac{c(0)}{c(z)} \quad (B-5)$$

and the transport equation,

$$2 \underline{k} \cdot \nabla A + (\nabla \cdot \underline{k}) A = 0 \quad . \quad (B-6)$$

The former equation can be solved by the method of characteristics (see Bleistein, 1984) which reduces the problem to the solution of a system of ordinary differential equations. The first of these equations are:

$$\frac{d\underline{x}}{d\sigma} = \underline{K} \quad , \quad \frac{dz}{d\sigma} = k_3 \quad ; \quad \underline{K} = (k_1, k_2) \quad , \quad \underline{k} = (\underline{K}, k_3) \quad (B-7)$$

$$\frac{d\underline{K}}{d\sigma} = 0 \quad , \quad \frac{dk_3}{d\sigma} = -n \, n'(z) \quad (B-8)$$

which define the rays, σ being the ray parameter. The source term in (B-2) makes

$$\underline{x}(0) = \underline{\xi} \quad , \quad z(0) = 0 \quad , \quad (B-9)$$

a natural choice as initial data for (B-7). The data for $\underline{k}(0)$ consists of an arbitrary unit vector (cf. (B-5), noting that $n(0) = 1$). To be specific we choose

$$\underline{K}(0) \text{ arbitrary} \quad , \quad k_3(0) = \sqrt{1-K^2} \quad (B-10)$$

where we have introduced

$$K = |\underline{K}| = \sqrt{k_1^2 + k_2^2} \quad . \quad (B-11)$$

From (B-10) and the fact that we are not considering turned rays here, it follows that $\underline{K} = (k_1, k_2)$ can be viewed as the direction numbers of the rays initiating from $(\xi, \eta, 0)$.

Using (B-6) and (B-7) we find

$$\frac{d\tau}{d\sigma} = \frac{n^2}{c(0)} \quad (B-12)$$

$$2 \frac{dA}{d\sigma} + (\nabla \cdot \underline{k}) A = 0 \quad (B-13)$$

which together with the data (B-3) completes the specification of g , expressed in (B-1), in terms of a system of ordinary differential equations.

Proceeding to the analysis, we first note that by (B-8), $\underline{K} = \underline{K}(0)$, so henceforth we simply write \underline{K} for this constant vector. Then the eikonal equation (B-5) gives us

$$k_z(K, z) = \sqrt{n^2 - K^2} \quad (B-14)$$

Then (B-7) yields

$$\frac{d\underline{x}}{dz} = \frac{\underline{K}}{\sqrt{n^2 - K^2}} \quad (B-15)$$

or

$$\underline{x} - \underline{\xi} = \underline{K} E(K, z) \quad (B-16)$$

where E is defined by (A-4). Similarly, we find

$$\tau = \frac{1}{c(0)} G(K, z) \quad (B-17)$$

where G is defined by (A-6).

If we introduce the ray Jacobian,

$$J = \frac{\partial(\underline{x}, z)}{\partial(\underline{k}, \sigma)} \quad (\text{B-18})$$

and use the fact that

$$\frac{dJ}{d\sigma} = J \nabla \cdot \underline{k} \quad (\text{B-19})$$

then the transport equation (B-6) can be recast as

$$A \sqrt{J} = \text{constant} \quad (\text{B-20})$$

Thus

$$A \sqrt{J} = \lim_{R \rightarrow 0} A \sqrt{J} \quad (\text{B-21})$$

which by (B-3) implies that

$$A = \frac{1}{\sqrt{J}} \lim_{R \rightarrow 0} \left[\frac{\sqrt{J}}{4\pi R} \right] \quad (\text{B-22})$$

We now indicate how to obtain the partial derivatives which are the elements of J .

First we take the reciprocal of the second equation in (B-7) and integrate to get

$$\sigma = \int_0^z \frac{dz'}{k_3(K, z')} .$$

We think of $z = z(\underline{K}, \sigma)$, differentiate this expression for σ and use $\partial\sigma/\partial k_j = 0$ to obtain

$$\frac{\partial z}{\partial k_j} = - k_j k_3 F \quad (B-23)$$

Similarly from (B-16), and using (B-23), one obtains

$$\frac{\partial x_i}{\partial k_j} = E \delta_{ij} , \quad j = 1, 2 \quad (B-24)$$

Since $\partial \underline{x}/\partial \sigma$ and $\partial z/\partial \sigma$ are given directly by (B-7), we are now able to form J . A short calculation, involving the use of (A-10) yields

$$J = k_3 E H . \quad (B-25)$$

It is easy to show that as $R \rightarrow 0$ (equivalently $\sigma \rightarrow 0$).

$$J \rightarrow \frac{z^2}{k_z^3} \quad . \quad (B-26)$$

In the same limit, (B-14) implies that

$$k_z \rightarrow k_z(K, 0) = \sqrt{1-K^2} = z/R \quad (B-27)$$

so that

$$J \rightarrow \frac{R^3}{z} = \frac{R^2}{k_z(K, 0)} \quad . \quad (B-28)$$

Then (B-22) gives

$$A = \frac{1}{\sqrt{k_z(K, z)E(K, z)H(K, z)}} \cdot \frac{1}{4\pi \sqrt{k_z(K, 0)}} \quad . \quad (B-29)$$

Since A and τ depend only on K, we need only employ the magnitude of (B-16) in the sequel:

$$|\underline{x} - \underline{\xi}| = KE(K, z) \quad . \quad (B-30)$$

Equation (B-14), (B-17), (B-29) and (B-30) are equivalent to equations (3-8) of the text.

Finally, we relate our parameter σ along the rays to the corresponding arclength parameter. From (B-7) we have

$$\frac{d\underline{x}}{d\sigma} \cdot \frac{d\underline{x}}{d\sigma} + \frac{dz}{d\sigma} \frac{dz}{d\sigma} = K^2 + k_z^2 = n^2 \quad (B-31)$$

and so

$$\frac{ds}{d\sigma} = n(z) \quad . \quad (B-32)$$

Using the second equation in (B-7) once more, we find

$$\frac{ds}{dz} = \frac{n}{k_z} \quad (B-33)$$

or

$$s = \int_0^z \frac{n(\zeta) d\zeta}{k_z(K, \zeta)} \quad . \quad (B-34)$$

APPENDIX C KIRCHHOFF DATA

The derivation presented in Cohen and Bleistein [1983] applies here with the constant c Green's function used there being replaced with the $c(z)$ Green's function derived in Appendix B. Thus, equation (8) of that paper may be recast in our present notation as

$$u_S(\underline{z}, \omega) \sim \int_S dS R \frac{\partial}{\partial n} g^2(\underline{x}, z; \underline{\xi}; \omega) \quad (C-1)$$

with g defined by equations (3-6) and the reflection coefficient R being defined by

$$R = \frac{\gamma - \gamma_1}{\gamma + \gamma_1} \quad (C-2)$$

where

$$\gamma = \hat{n} \cdot \nabla \tau, \quad \gamma_1 = \text{sgn}(\gamma) \sqrt{\gamma^2 + \frac{1}{c_1^2} - \frac{1}{c^2}}, \quad (C-3)$$

and \hat{n} is the upward normal to S . Here c_1 is defined by (31).

Since g has the form given in (B-1)

$$\frac{\partial}{\partial n} g^2 \sim 2i\omega \hat{n} \cdot \nabla \tau g^2, \quad (C-4)$$

subject to

$$|\underline{x} - \underline{\xi}| = KE(K, z) \quad . \quad (C-5)$$

Hence (B-1), with the definition of γ given in (C-3) and that of S given by (29), yields

$$\frac{\partial}{\partial n} g^2 \sim 2i\omega\gamma S e^{2i\omega\tau} \quad . \quad (C-6)$$

Thus (26) follows from the form, (25), of the surface. It remains to establish the detailed calculation of γ given in (28).

We have:

$$\begin{aligned} \gamma &= \hat{n} \cdot \nabla \tau \\ &= \frac{1}{c(0)} \hat{n} \cdot \nabla G(K, z) \\ &= \frac{1}{c(0)} \frac{(h', 0, -1)}{\sqrt{1+(h')^2}} \cdot \left[G_K \frac{\partial K}{\partial x}, G_K \frac{\partial K}{\partial y}, G_K \frac{\partial K}{\partial z} + G_z \right] \\ &= \frac{1}{c(0)} \frac{(h', 0, -1)}{\sqrt{1+(h')^2}} \cdot \left[h' G_K \frac{\partial K}{\partial x} - G_K \frac{\partial K}{\partial z} - G_z \right] \quad . \end{aligned} \quad (C-7)$$

From the constraint (C-5) we compute

$$\frac{\partial K}{\partial x} = \frac{x - \xi}{\rho(K E)_K}, \quad \frac{\partial K}{\partial z} = - \frac{K E_z}{(K E)_K} \quad (C-8)$$

and then (28) follows from (C-7), (C-8) and the results of Appendix A.

APPENDIX D STATIONARY PHASE CALCULATIONS

Assuming that the phase, $\bar{\Phi}(\underline{x})$, has a single simple stationary point \underline{x}_s ,

$$\nabla \bar{\Phi}(\underline{x}_s) = \underline{0} \quad , \quad \det \frac{\partial^2 \bar{\Phi}}{\partial x_i \partial x_j}(\underline{x}_s) \neq 0 \quad , \quad (D-1)$$

the integral,

$$I(\lambda) \sim \int e^{i\lambda \bar{\Phi}(\underline{x})} A(\underline{x}) d^n \underline{x} \quad , \quad |\lambda| \gg 1 \quad (D-2)$$

can be evaluated asymptotically by the multidimensional stationary phase formula, (see Bleistein [1984] or Bleistein and Handelsman [1975]).

$$I(\lambda) \sim \left[\frac{2\pi}{|\lambda|} \right]^{n/2} \frac{A_s}{\sqrt{|\det(\bar{\Phi}_{ij})|}} \exp\{i\lambda \bar{\Phi}_s + i \frac{\pi}{4} \text{sgn } \lambda \text{ sig}(\bar{\Phi}_{ij})\} \quad , \quad (D-3)$$

Here A_s , $\bar{\Phi}_s$, and $(\bar{\Phi}_{ij})$, denote respectively the amplitude A , the phase $\bar{\Phi}$ and the Hessian matrix, $(\partial^2 \bar{\Phi} / \partial x_i \partial x_j)$ evaluated at the stationary point, $\underline{x} = \underline{x}_s$. Further, $\text{sig}(\bar{\Phi}_{ij})$ denotes the signature of $(\bar{\Phi}_{ij})$, i.e. the number of positive eigenvalues minus the number of negative eigenvalues.

If there were several stationary points, (D-3) would be replaced by a sum over the contributions from these several points. If the Hessian matrix vanished, (D-3) would have to be replaced by a more general result. However, only (D-3) is required here.

In the second application of (D-5) to follow, $n = 4$. Although the evaluation of the signature of a symbolic 4×4 matrix can be tedious or

impossible, our task is considerably simplified by the fact that our Hessian matrices have the special form,

$$\bar{H} = \begin{bmatrix} \alpha & 0 & v & 0 \\ 0 & \beta & 0 & u \\ v & 0 & \gamma & 0 \\ 0 & u & 0 & \delta \end{bmatrix} , \quad (D-4)$$

whence

$$\det \bar{H} = (\alpha\gamma - v^2)(\beta\delta - u^2) . \quad (D-5)$$

To evaluate the signature, we need to determine the roots, σ , of the eigenvalue equation,

$$\det (\bar{H} - \sigma I) = 0 , \quad (D-6)$$

where I is the identity matrix. From (D-5), we find at once:

$$\det(\bar{H} - \sigma I) = \left[\sigma^2 - (\alpha + \gamma) \sigma + \alpha\gamma - v^2 \right] \left[\sigma^2 - (\beta + \delta) \sigma + \beta\delta - u^2 \right] . \quad (D-7)$$

Thus,

$$v^2 > \alpha\gamma , \quad u^2 > \beta\delta \Rightarrow \text{sig} (\bar{H}_{ij}) = 0 . \quad (D-8)$$

and similarly,

$$v^2 > \alpha\gamma , \quad u^2 < \beta\delta \Rightarrow \text{sig} (\bar{H}_{ij}) = \pm 2 , \quad (D-9)$$

etc.

We now turn to the specific stationary phase calculations in the text. We shall freely use the results of Appendix A without explicit citation.

First we examine the phase in (12):

$$\Phi(\xi) = G(K, z) - G(K', z') = c(0) [\tau(K, z) - \tau(K', z')] \quad (D-10)$$

where G , K , and K' are defined by:

$$G(K, z) = \int_0^z \frac{n^2(\xi)}{k(\xi, K)} d\xi ,$$

$$|\underline{x} - \underline{\xi}| = KE(K, z) \equiv KE , \quad (D-11)$$

$$|\underline{x}' - \underline{\xi}| = K'E(K', z') \equiv K'E' . \quad (D-12)$$

Variables \underline{x} , z , \underline{x}' , and $z' \neq z$ are viewed as fixed for this calculation. Implicit differentiation of (D-11) with respect to ξ_i yields (after appealing to Appendix A),

$$\frac{\xi_i - x_i}{\rho} = (KE)_K \frac{\partial K}{\partial \xi_i} = H \frac{\partial K}{\partial \xi_i} ;$$

so

$$\frac{\partial K}{\partial \xi_i} = \frac{\xi_i - x_i}{KEH} , \quad \frac{\partial K'}{\partial \xi_i} = \frac{\xi_i - x'_i}{K'E'H'} .$$

Using these results we have

$$\bar{\Phi}_i = \frac{\partial \bar{\Phi}}{\partial \xi_i} = G_K \left[\frac{\partial K}{\partial \xi_i} - \frac{\partial K'}{\partial \xi_i} \right] = \frac{\xi_i - x_i}{E} - \frac{\xi_i - x'_i}{E'} \quad . \quad (D-13)$$

To obtain our stationarity condition we set $\bar{\Phi}_i = 0$ and have

$$\frac{\xi_i - x_i}{E} = \frac{\xi_i - x'_i}{E'} \quad ; \quad i = 1, 2 \quad .$$

Using this result with (D-11) and (D-12) leads to

$$K = K' \quad . \quad (D-14)$$

This condition says that the critical value of ξ is such that the points $\underline{r} = (\underline{x}, z)$ and $\underline{r}' = (\underline{x}', z')$ lie on the same ray emanating from $(\xi, 0)$. See Fig. 1 above. We symbolically denote this stationarity choice of ξ by ξ_s .

We now proceed to calculate the Hessian matrix $(\bar{\Phi}_{ij})$. Differentiating expression (D-13) with respect to ξ_j leads to

$$\begin{aligned} \bar{\Phi}_{ij} &= \frac{\partial^2 \bar{\Phi}}{\partial \xi_i \partial \xi_j} = \delta_{ij} \left[\frac{1}{E} - \frac{1}{E'} \right] - k_i k_j \left[\frac{F}{EH} - \frac{F'}{E'H'} \right] \\ &\equiv \delta_{ij} \varepsilon - k_i k_j \gamma \quad . \end{aligned}$$

We have again appealed to results in Appendix A. It now follows that

$$\begin{aligned}
\det (\bar{\Phi}_{ij}) &= \det \begin{bmatrix} \varepsilon - k_1^2 \gamma & -k_1 k_2 \gamma \\ -k_1 k_2 \gamma & \varepsilon - k_2^2 \gamma \end{bmatrix} \\
&= \varepsilon^2 - \varepsilon K^2 \gamma = \varepsilon (\varepsilon - K^2 \gamma) \\
&= \left[\frac{1}{E} - \frac{1}{E'} \right] \left[\frac{1}{E} - \frac{1}{E'} - K^2 \left[\frac{F}{EH} - \frac{F'}{E'H'} \right] \right] \\
&= \left[\frac{1}{E} - \frac{1}{E'} \right] \left[\frac{1}{E} - \frac{1}{E'} - \left[\frac{H-E}{EH} - \frac{H'-E'}{E'H'} \right] \right] \\
&= \left[\frac{1}{E} - \frac{1}{E'} \right] \left[\frac{1}{H} - \frac{1}{H'} \right] \\
&= \frac{E' - E}{EE'} \frac{H' - H}{HH'} .
\end{aligned}$$

Note $\det(\bar{\Phi}_{ij}) > 0$ when $z \neq z'$ since E and H are z integrals with positive integrands (see Appendix A). Similarly, in expanding $\det(\bar{\Phi}_{ij} - \sigma I)$ one finds eigenvalues

$$\sigma_1 = \frac{E' - E}{EE'} , \quad \sigma_2 = \frac{H' - H}{HH'}$$

and therefore concludes that $\text{sig}(\bar{\Phi}_{ij}) = 2 \text{sgn}(z'-z)$. Note that

$$e^{i \frac{\pi}{4} \text{sig}(\bar{\Phi}_{ij})} = e^{i \frac{\pi}{2} \text{sgn}(z'-z)} = i \text{sgn}(z'-z) .$$

Using these results in (D-3) leads to the following asymptotic (large ω) approximation to the ξ integral in (12):

$$\frac{B(\underline{x}', \underline{\xi}) e^{2i\omega[\tau(K, z) - \tau(K, z')]} }{c^2(z) k_j(K, 0) k_j(K, z) E(K, z) H(K, z)} \cdot \frac{i\pi c(0) \operatorname{sgn}(z' - z)}{\omega} \frac{(EE'HH')^{1/2}}{[(E - E')(H - H')]^{1/2}} .$$

We now turn to the stationary phase evaluation of (32) in the four variables $\bar{x}, \bar{y}, \bar{\xi}, \bar{\eta}$. Here the phase is

$$\Psi = G(\bar{K}, \bar{z}) - G(K, z) = c(0)[\tau(\bar{K}, \bar{z}) - \tau(K, z)] \quad (D-15)$$

subject to the usual constraints

$$|\underline{x} - \underline{\xi}| = KE(K, z) \quad , \quad |\bar{x} - \bar{\xi}| = \bar{K}E(\bar{K}, \bar{z}) \quad . \quad (D-16)$$

This stationary phase evaluation is somewhat more difficult because \bar{z} depends on \bar{x} :

$$\bar{z} = h(\bar{x}) \quad , \quad (D-17)$$

Implicit differentiation in \bar{x} , of the second equation in (D-16) leads to

$$\frac{\bar{x} - \bar{\xi}}{\bar{\rho}} = (\bar{K}E)_K \frac{\partial \bar{K}}{\partial \bar{x}} + \bar{K}E_z = \bar{H} \frac{\partial \bar{K}}{\partial \bar{x}} + \frac{\bar{K}}{\bar{k}_z} h' \quad , \quad (D-18)$$

so

$$\frac{\partial \bar{K}}{\partial \bar{x}} = \frac{\bar{x} - \bar{\xi}}{\bar{K}E\bar{H}} - \frac{\bar{K}h'}{\bar{k}_z H} \quad (D-19)$$

where we use notations like

$$\bar{E} = E(\bar{K}, \bar{z}) \quad (D-20)$$

for quantities which depend on \bar{K} and \bar{z} . The remaining partials of \bar{K} and K are simpler since \bar{z} does not depend on \bar{y} , ξ , or y , and since z is a completely independent variable:

$$\frac{\partial \bar{K}}{\partial \bar{y}} = \frac{y-\eta}{\bar{K}\bar{E}\bar{H}} \quad , \quad \frac{\partial \bar{K}}{\partial \xi} = \frac{\xi-\bar{x}}{\bar{K}\bar{E}\bar{H}} \quad , \quad \dots \quad (D-21)$$

These results allow one to compute:

$$\begin{aligned} \bar{\Phi}_{\bar{x}} &= \bar{G}_{\bar{K}} \frac{\partial \bar{K}}{\partial \bar{x}} + \bar{G}_{\bar{z}} h'(\bar{x}) \\ &= \frac{\bar{x}-\xi}{\bar{E}} - \frac{\bar{K}^2}{\bar{k}_z} h' + \frac{n^2(\bar{z})}{\bar{k}_z} \\ &= \frac{\bar{x}-\xi}{\bar{E}} + \bar{k}_z h' \quad , \\ \bar{\Phi}_{\bar{y}} &= \frac{\bar{y}-\eta}{\bar{E}} \quad , \quad \bar{\Phi}_{\xi} = \frac{\xi-\bar{x}}{\bar{E}} - \frac{\xi-x}{E} \quad , \quad \bar{\Phi}_{\eta} = \frac{n-\bar{y}}{\bar{E}} - \frac{\eta-y}{E} \quad . \end{aligned} \quad (D-22)$$

The \bar{y} , η derivatives give rise to the simple stationarity conditions:

$$\bar{y} = \eta = y \quad . \quad (D-23)$$

Hence the constraints (D-16) reduce to

$$|x - \xi| = KE(K, z) \quad , \quad |\bar{x} - \xi| = \bar{K}E(\bar{K}, \bar{z}) \quad , \quad (D-24)$$

which can be rewritten as

$$x - \xi = \mu KE \quad , \quad \mu = \text{sgn}(x - \xi) \quad (D-25)$$

and

$$\bar{x} - \xi = \bar{\mu} \bar{K} \bar{E} \quad , \quad \bar{\mu} = \text{sgn}(\bar{x} - \xi) \quad . \quad (\text{D-26})$$

These results allow us to simplify the remaining partials in (D-22) to

$$\bar{\Phi}_{\bar{x}} = \bar{\mu} \bar{K} + \bar{k}_z h' \quad (\text{D-27})$$

and

$$\bar{\Phi}_{\bar{y}} = -\bar{\mu} \bar{K} + \mu K \quad . \quad (\text{D-28})$$

For stationarity, we must have

$$\mu = \bar{\mu} = -\text{sgn}(h') \quad (\text{D-29})$$

and then also

$$K = \bar{K} = \bar{k}_z h' \quad . \quad (\text{D-30})$$

The latter equality allows us to determine that

$$\bar{K} = \frac{n(\bar{z}) |h'|}{\sqrt{1+h'^2}} \quad , \quad \bar{k}_z = \frac{n(\bar{z})}{\sqrt{1+h'^2}} \quad . \quad (\text{D-31})$$

Finally (D-29) allows us to restate (D-25), (D-26) as

$$x - \xi = -\bar{K} E(\bar{K}, z) \text{sgn}(h') \quad (\text{D-32})$$

and

$$\bar{x} - \xi = -\bar{K} E(\bar{K}, \bar{z}) \text{sgn}(h') \quad . \quad (\text{D-33})$$

At the stationary point, the phase is

$$\bar{\Phi}_s = G(\bar{K}, \bar{z}) - G(\bar{K}, z) \quad . \quad (D-34)$$

Further, a number of terms of the Hessian matrix vanish because of (D-23) leaving us with

$$(\bar{\Phi}_{ij}) = \begin{bmatrix} \bar{\Phi}_{\bar{x}\bar{x}} & 0 & \bar{\Phi}_{\bar{x}\xi} & 0 \\ 0 & \bar{\Phi}_{\bar{y}\bar{y}} & 0 & \bar{\Phi}_{\bar{y}n} \\ \bar{\Phi}_{\bar{x}\xi} & 0 & \bar{\Phi}_{\xi\xi} & 0 \\ 0 & \bar{\Phi}_{\bar{y}\eta} & 0 & \bar{\Phi}_{\eta\eta} \end{bmatrix} \Big|_s \quad (D-35)$$

which has the form (D-4) with

$$v^2 - \alpha\gamma = (\bar{\Phi}_{\bar{x}\xi}^2 - \bar{\Phi}_{\bar{x}\bar{x}} \bar{\Phi}_{\xi\xi}) \Big|_s \quad , \quad (D-36)$$

and

$$u^2 - \beta\gamma = (\bar{\Phi}_{\bar{y}\eta}^2 - \bar{\Phi}_{\bar{y}\bar{y}} \bar{\Phi}_{\eta\eta}) \Big|_s \quad . \quad (D-37)$$

We observe that (D-19) and the stationarity conditions yield

$$\frac{\partial \bar{K}}{\partial \bar{x}} \Big|_s = \frac{\bar{\mu}}{\bar{H}} - \frac{h' |h'|}{\bar{H}} = \frac{\bar{\mu}}{\bar{H}} (1+h'^2) \quad , \quad \bar{\mu} = -\text{sgn}(h') \quad (D-38)$$

and similarly

$$\frac{\partial \bar{K}}{\partial \xi} \Big|_s = -\frac{\bar{\mu}}{\bar{H}} \quad , \quad \frac{\partial K}{\partial \xi} \Big|_s = \frac{-\bar{\mu}}{H(\bar{K}, z)} \quad . \quad (D-39)$$

These facts allow to evaluate (D-36) as

$$v^2 - \alpha\gamma = \frac{(1+h'^2)^2}{H(\bar{K}, z)H(\bar{K}, z)} + \frac{n(\bar{z})h''}{\sqrt{1+h'^2}} \left[\frac{1}{H(\bar{K}, z)} - \frac{1}{H(\bar{K}, \bar{z})} \right] \quad (D-40)$$

and (D-37) as

$$u^2 - \beta\gamma = \frac{1}{E(\bar{K}, z)E(\bar{K}, \bar{z})} > 0 \quad (D-41)$$

Thus, "near" the reflector, $z = \bar{z}$, both $v^2 > \alpha\gamma$, $u^2 > \beta\delta$ hold and from (D-8) we have $\text{sig}(\Phi_{ij}) = 0$. Finally, applying all these results to the integral given by (32) leads to the approximation (40).

APPENDIX E A 1-DIMENSIONAL DISTRIBUTION

In (19) we were lead to the singular form

$$\frac{\delta'(\tau-\tau')}{z-z'} = \frac{1}{z-z'} \frac{d}{d\tau} \delta(\tau-\tau') \quad . \quad (E-1)$$

This requires some careful interpretation since as $z \rightarrow z'$, $\tau \rightarrow \tau'$ and hence the form is very singular at the point at which the "action" takes place. We here first show that, with

$$\rho \equiv |\underline{r}-\underline{r}'| \operatorname{sgn} (z-z') \quad , \quad (E-2)$$

we have

$$\lim_{z \rightarrow z'} \frac{\rho}{z-z'} = \frac{n(z')}{k_3(K, z')} \quad , \quad (E-3)$$

$$\lim_{z \rightarrow z'} \frac{\rho}{\tau-\tau'} = \lim_{\tau \rightarrow \tau'} \frac{\rho}{\tau-\tau'} = c(z') \quad . \quad (E-4)$$

To establish (E-3), we appeal to (B-16) and note that since \underline{r} and \underline{r}' are on the same ray:

$$\underline{r} - \underline{r}' = (\underline{x}, z) - (\underline{x}', z') = \left[\underline{K} \int_{z'}^z \frac{d\underline{r}}{k_1(K, \underline{r})} , z - z' \right] .$$

Hence as $z \rightarrow z'$:

$$\begin{aligned} \frac{|\underline{r} - \underline{r}'|}{|z - z'|} &= \frac{1}{|z - z'|} \left[K^2 \left[\int_{z'}^z \frac{d\underline{r}}{k_1(K, \underline{r})} \right]^2 + (z - z')^2 \right]^{1/2} \\ &\rightarrow \left[\frac{K^2}{n^2(z') - K^2} + 1 \right]^{1/2} = \frac{n(z')}{k_1(K, z')} . \end{aligned} \quad (E-5)$$

Using this result and (E-2) establishes (E-3).

To obtain (E-4) we use (A-6) to get

$$\begin{aligned} \frac{\tau - \tau'}{z - z'} &= \frac{1}{c(0)(z - z')} \int_{z'}^z \frac{n^2(\underline{r}) d\underline{r}}{k_1(K, \underline{r})} \\ &\rightarrow \frac{n^2(z')}{c(0)k_1(K, z')} \text{ as } z \rightarrow z' . \end{aligned}$$

Using this together with (E-3) establishes (E-4).

Applying (E-4) we have

$$\begin{aligned}
 \delta(\tau-\tau') &= \delta[\rho/c(z')] = c(z') \delta(\rho) , \\
 \frac{d}{d\tau} \delta(\tau-\tau') &= \frac{d\rho}{d\tau} \frac{d}{d\rho} \delta(\tau-\tau') = c(z') \frac{d}{d\rho} [c(z')\delta(\rho)] \\
 &= c^2(z') \frac{d}{d\rho} \delta(\rho) = c^2(z')\delta'(\rho) .
 \end{aligned}
 \tag{E-6}$$

This result, (E-6), and (E-3) are combined in the main text to further interpret the singular expression (E-1).

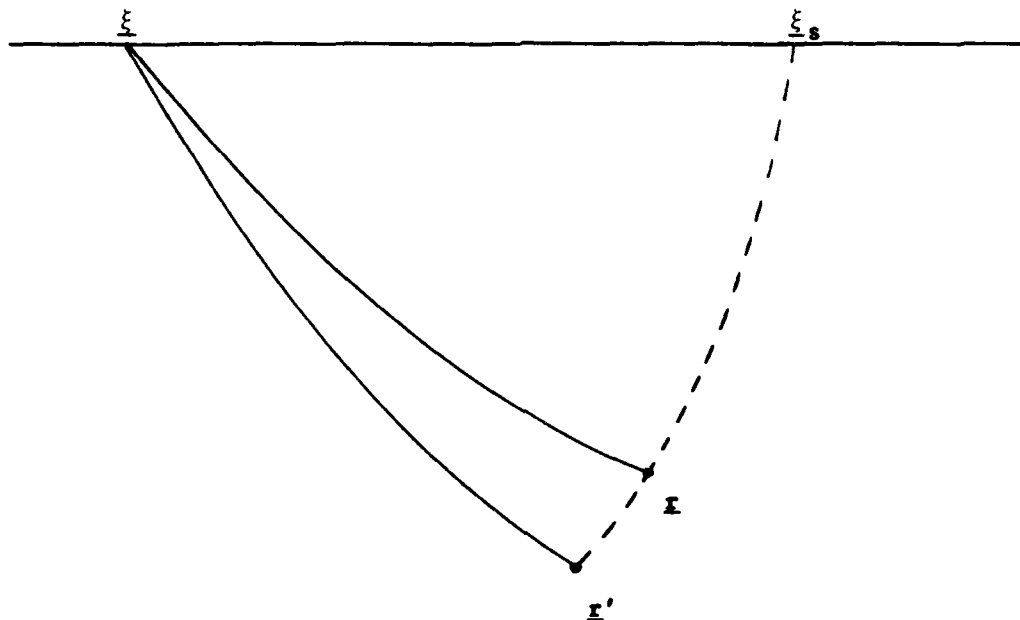


Figure 1: Two rays (solid lines) connecting an arbitrary surface point ξ to r and r' ; and the single ray emanating from point ξ_s determined by stationary phase.

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ABSTRACT

The purpose of this work is to present an inversion algorithm for backscattered ("stacked") seismic data which will reconstruct the velocity profile in realistic earth conditions. The basic approach follows that of the original Cohen and Bleistein paper [1979a] in that high frequency asymptotics and perturbation methods are used. However, in the original paper the perturbation was relative to a constant reference speed, whereas the current work uses a reference speed which may vary with depth. This greatly enhances the validity of the perturbation assumption and hence the inversion results. On the other hand, the new algorithm enjoys the same economies and stability properties of the original algorithm, making it very competitive with current migration schemes.

Four major assumptions are made: (i) the acoustic wave equation is an adequate model, (ii) stacked data has amplitude information worth preserving fairly accurately, (iii) the actual reflectivity coefficients can be adequately modeled as perturbations from a continuous reference velocity which depends only on the depth variable, and (iv) the subsurface can be adequately modeled as a series of layers with jump discontinuities in the velocity (or impedance) at these layers.

While the algorithm is particularly suited for data generated by a number of reflecting surfaces, its validity for a single reflector is demonstrated by applying the algorithm to Kirchhoff data for a quite general surface.

A key feature of the approach of this paper is the repeated application of high frequency asymptotic methods; both in obtaining the basic integral equation describing the unknown velocity correction, and in the inversion of this integral equation. Perhaps a noteworthy feature is that the underlying integral equation is in the form of a generalized Fourier integral equation; and the method for its (approximate) inversion may prove to be applicable to a wide range of such problems.

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